SURFACE AREA. II

BY
HERBERT FEDERER

1. Introduction. All definitions of §2 of our preceding paper, *Surface Area*. I(1), are again in force.

Throughout the present paper we fix two positive integers $m \leqq n$. We are interested in the validity of the formula

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx$$

where f is a function on E_m to E_n and $T \subset E_m$. The left-hand member of this equation is the integral, relative to our m-dimensional surface measure over E_n, of the multiplicity of f on T. We believe that this integral appropriately measures the area of that part of the surface f which corresponds to T. The integrand of the right-hand member is the Jacobian associated with f by means of its approximate differential.

Our main results are contained in the Theorems 5.1, 5.2, 6.1 and 7.2. The last of these, which is a specialization of the preceding theory, yields new information about Dini derivatives.

In case $m=2$ it follows from the work of Radó, McShane, and Morrey(2) that the right-hand member of the identity frequently equals the Lebesgue surface area of f.

If f is univalent, the left-hand integral is the Φ measure of the image of T under f.

2. Definition of the measure Φ.

2.1 *Definition.* We say P is a *projecting function* (on E_n to E_m) if and only if there is an orthogonal transformation R on E_n to E_n such that the relations

$$R(x) = y \quad \text{and} \quad P(x) = (y_1, y_2, \cdots, y_m)$$

are equivalent whenever x and y are points in E_n.

For $S \subset E_n$ we define $\gamma(S)$ to be the supremum of numbers of the form $|P^*(S)|$ where P is a projecting function(3).

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(1) Trans. Amer. Math. Soc. vol. 55 (1944) pp. 420–437. We hereafter refer to this paper as SA I.

(2) T. Radó, *Über das Flächenmass rektifizierbarer Flächen*, Math. Ann. vol. 100, p. 445; *On the derivative of the Lebesgue area of continuous surfaces*, Fund. Math. vol. 30, p. 34.

E. J. McShane, *Integrals over surfaces in parametric form*, Ann. of Math. vol. 34, p. 815.

C. B. Morrey, *A class of representations of manifolds*, Amer. J. Math. vol. 55, p. 683.

(3) $|P^*(S)|$ is the m-dimensional Lebesgue measure of $P^*(S)$.

Whenever $S \subset E_n$ and $r > 0$, we denote by $\gamma_r(S)$ the infimum of numbers of the form

$$\sum_{T \in F} \gamma(T)$$

where F is a countable family of open connected subsets of E_n, each of diameter less than r, such that $S \subset \sigma(F)$.

We now define the function Φ on the set of all subsets of E_n by the relation

$$\Phi(S) = \lim_{r \to 0+} \gamma_r(S).$$

This limit exists because $\gamma_r(S)$ is monotone in r for each fixed set S.

2.2 *Remark.* If $m = 2$ and $n = 3$, then the functions γ, γ_r, Φ of Definition 2.1 are identical with the functions bearing the same names in SA I.

In fact the reader will easily convince himself that the functions P_c introduced in §3 of SA I are projection functions in the sense of 2.1 of this paper; and each of our present projection functions is of the form $R:P_c$ where R is an orthogonal transformation of E_2. Using the invariance of plane Lebesgue measure under such transformations it may be seen that the old and new meanings of γ are identical; consequently the same is true for γ_r and Φ.

2.3 *Remark.* Theorems 3.4, 3.5, 3.6, 4.4 of SA I are true in our present general setting with P_c replaced by any projecting function P. The proofs are exactly the same as before.

2.4 *Remark.* If $m = n$, then Φ is m-dimensional Lebesgue measure.

2.5 *Remark.* If $m = 1$, then Φ is Carathéodory linear measure over E_n. In fact $\gamma(T) = \operatorname{diam} T$ for every connected set $T \subset E_n$.

3. Further definitions.

3.1 *Definition.* We write $\det A$ for the determinant of a square matrix A.

Now suppose p and q are positive integers.

A function L on E_p to E_q is said to be *linear* if and only if L is continuous and

$$L(x + y) = L(x) + L(y) \quad \text{for} \quad x \in E_p,\ y \in E_p.$$

We make no distinction between L and its matrix. The jth column ($j = 1, 2, \cdots, p$) of L is a point of E_q and denoted by L^j. We further define the *norm* of L by the relation

$$\|L\| = \sup_{|x|=1} |L(x)|.$$

If $p \leqq q$, then

$$\Delta(L) = \left\{ \sum_{x \in S} (\det M_x)^2 \right\}^{1/2}$$

where $x \in S$ if and only if x is a set of p integers between 1 and q; and M_x is the minor of L which is made up from the p rows whose indices are elements of x.

3.2 *Definition.* Suppose g is a numerically valued function on E_p and $x \in E_p$. Then we define

$$\limsup_{z \to x} \text{ap}\, g(z)$$

as the infimum of numbers of the form

$$\limsup_{z \to x, z \in A} g(z)$$

where A is a Lebesgue measurable subset of E_p with density 1 at x.

3.3 *Definition.* Suppose f is a function on E_p to E_q.

We say f is *differentiable at* x if and only if there is a linear function L on E_p to E_q such that

$$\limsup_{z \to x} \frac{| f(z) - f(x) - L(z - x) |}{| z - x |} = 0.$$

The (unique) linear function occurring in this definition is called the *differential of* f *at* x.

We say f is *approximately differentiable at* x if and only if there is a linear function L on E_p to E_q such that

$$\limsup_{z \to x} \text{ap} \frac{| f(z) - f(x) - L(z - x) |}{| z - x |} = 0.$$

The (unique) linear function occurring in this definition is called the *approximate differential of* f *at* x.

If $p \leq q$ then we define the function Jf as follows: The domain of Jf is the set of all points of approximate differentiability of f and for each such point x we let

$$Jf(x) = \Delta(L)$$

where L is the approximate differential of f at x.

3.4 *Remark.* Suppose f is a function on E_m to E_n.

It may interest the reader, although we shall not use it, that in case $m = 2$ our definitions imply

$$Jf = (EG - F^2)^{1/2}$$

where the right-hand member has its classical meaning.

Furthermore in case $m = n$ our $Jf(x)$ equals the absolute value of the classical Jacobian determinant of f at x.

Similarly in case $m = 1$ our $Jf(x)$ is the absolute value of the approximate derivative of f at x.

3.5 *Remark.* If f is a univalent differentiable function on E_m to E_n, then f transforms E_m into a Riemannian manifold R_m. Fix a point $x \in E_m$ and let g be the matrix of the fundamental differential quadratic form associated with the point $f(x) \in R_m$ and the coordinate system defined by f. Denoting the differential of f at x by L and its conjugate by $\bar{L}$, we have $g=\bar{L}:L$ and see from the proof of Lemma 4.2 below that $(\det g)^{1/2}=Jf(x)$.

No use is made of this fact in this paper.

4. Lipschitzian surfaces.

4.1 Lemma. *If f is a function on $T\subset E_m$ to E_n and M is a number such that*

$$|f(z)-f(x)|\leqq M|z-x| \quad \textit{whenever} \quad x\in T, z\in T$$

then

$$\Phi[f^*(T)]\leqq (5M)^m|T|.$$

Proof. Let $r>0$.

Let S be an open set for which $T\subset S$ and $|S|\leqq|T|+r$. Whenever $x\in T$, $\rho>0$ and

$$\alpha=E_m \underset{z}{E}[|z-x|\leqq\rho]$$

we denote

$$\tilde{\alpha}=E_m \underset{z}{E}[|z-x|<5\rho], \qquad \bar{\alpha}=E_n \underset{y}{E}[|y-f(x)|<5M\rho]$$

and readily check that

$$f^*(T\alpha)\subset\bar{\alpha}, \qquad \gamma(\bar{\alpha})=M^m|\tilde{\alpha}|=(5M)^m|\alpha|.$$

Now let F be the family of all closed spheres α with center in T and such that $\alpha\subset S$, diam $\bar{\alpha}<r$. Clearly F covers T and we use a well known covering theorem(4) to select such a disjointed subfamily G of F that

$$T\subset\sum_{\alpha\in G}\tilde{\alpha}.$$

Hence $f^*(T)\subset\sum_{\alpha\in G}\bar{\alpha}$ and

$$\gamma_r[f^*(T)]\leqq\sum_{\alpha\in G}\gamma(\bar{\alpha})=(5M)^m\sum_{\alpha\in G}|\alpha|\leqq(5M)^m|S|\leqq(5M)^m(|T|+r).$$

Let $r\to 0$.

4.2 Lemma. *If L is a linear function on E_m to E_n and R is an orthogonal transformation on E_n to E_n, then $\Delta(R:L)=\Delta(L)$.*

Proof. For each $n\times m$ matrix A we denote its conjugate $m\times n$ matrix by $\bar{A}$, that is $\bar{A}_i^j=A_j^i$; hence the Cauchy-Binet theorem(5) implies

$$[\Delta(A)]^2=\det(\bar{A}:A).$$

From this we infer

$$\begin{aligned}[\Delta(R:L)]^2&=\det[(\overline{R:L}):(R:L)]=\det[(\bar{L}:\bar{R}):(R:L)]\\&=\det[\bar{L}:(\bar{R}:R):L]=\det[(\bar{L}:L)]=[\Delta(L)]^2.\end{aligned}$$

(4) A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235, Theorem 3.5.

(5) A. C. Aitken, *Determinants and matrices*, Edinburgh, 1942, p. 86.

4.3 Lemma. *If f is a function on E_m to E_n; M is a number such that*

$$|f(z)-f(x)| \leqq M|z-x| \quad \text{for} \quad z \in E_m,\ x \in E_m;$$

A is a Lebesgue measurable subset of E_m; f is differentiable at each point of A and

$$0 < \lambda < Jf(x) < \mu < \infty \quad \text{for} \quad x \in A;$$

then

$$\lambda|A| \leqq \int N(f, A, y)d\Phi y \leqq \mu|A|.$$

Proof. Denoting the family of all Lebesgue measurable subsets of A by K, we infer from 4.1 that

$$f^*(X) \text{ is } \Phi \text{ measurable for } X \in K.$$

For $x \in A$ let $Df(x)$ be the differential of f at x. For $k=1, 2, 3, \cdots$ we define

$$\eta_k(x) = \sup_{0<|z-x|<k^{-1}} \frac{|f(z)-f(x)-[Df(x)](z-x)|}{|z-x|}$$

and are told by 3.3 that

$$\eta_k(x) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for each} \quad x \in A.$$

Helped by Egoroff's theorem we next select such disjoint closed subsets $B_1, B_2, B_3, \cdots$ of A that

(1) $$|A-B| = 0, \quad \text{where} \quad B = \sum_{s=1}^{\infty} B_s,$$

and

$$\eta_k(x) \to 0, \ \textit{uniformly for} \ x \in B_s, \quad \text{as} \quad k \to \infty$$

for each positive integer s. Thus f is uniformly differentiable on B_s. Using the standard theorem on uniform convergence we also note that the partial derivatives $D_1f, D_2f, \cdots, D_mf$ are continuous relative to B_s. In the terminology of Hassler Whitney(6) all this implies that *f is of class C' in B_s in terms of the functions f and $D_1f, D_2f, \cdots, D_mf$* for each positive integer s.

Now Whitney's extension theorem(7) gives us such functions $g_1, g_2, g_3, \cdots$ with continuous partial derivatives on E_m to E_n that

$$g_s(x) = f(x) \quad \text{and} \quad D_jg_s(x) = D_jf(x)$$

for $x \in B_s$; $s=1, 2, 3, \cdots; j=1, 2, \cdots, m$. Since

$$|D_jg_s(x)| = |D_jf(x)| \leqq M \quad \text{for} \quad x \in B_s$$

(6) H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. vol. 36 (1934) p. 64.

(7) Loc. cit. (6), p. 69, Lemma 2.

and $D_j g_s$ is continuous, we can select open sets $Q_1, Q_2, Q_3, \cdots$ such that $B_s \subset Q_s \subset E_m$ and

(2) $$|g_s(x) - g_s(z)| \leqq 2M|x - z| \quad \text{whenever } x \in Q_s,\ z \in Q_s.$$

The remainder of the proof is divided into three parts.

Part 1. If s is a positive integer, $x \in B_s$ and $r > 0$, then there is a family $F(x)$ of closed subsets of Q_s such that

$$0 = \inf_{\alpha \in F(x)} \operatorname{diam} \alpha; \qquad 0 < \inf_{\alpha \in F(x)} \frac{|\alpha|}{(\operatorname{diam} \alpha)^m};$$

$\alpha \in F(x)$ implies $x \in \alpha$, $\lambda|\alpha| < \Phi[g_s^*(\alpha)]$ and associated with α is an open sphere $\bar{\alpha} \subset E_n$ for which $g_s^*(\alpha) \subset \bar{\alpha}$, diam $\bar{\alpha} < r$ and $\gamma(\bar{\alpha}) < \mu|\alpha|$.

Proof. Abbreviate $L = Df(x)$. Select an orthogonal transformation R on E_n to E_n such that

(3) $$R(L^j) \in E_n \mathop{E}_{y} [y_i = 0 \text{ for } i = m + 1, m + 2, \cdots, n]$$

for $j = 1, 2, \cdots, m$; and let P be the projecting function such that the relations

$$R(y) = z \quad \text{and} \quad P(x) = (z_1, z_2, \cdots, z_m)$$

are equivalent whenever y and z are points of E_n. Take

$$p = P:g_s, \qquad T = P:L,$$

and note that L is the differential of g_s at x. Thus T is the differential of p at x and from (3) and Lemma 4.2 we infer

(4) $$Jp(x) = |\det T| = \Delta(T) = \Delta(P:L) = \Delta(R:L) = \Delta(L) = Jf(x) > 0.$$

Using (3) again we verify

(5) $$|T(w)| = |R[L(w)]| = |L(w)| \quad \text{for} \quad w \in E_m.$$

By virtue of (4) there exists an inverse U of T.

Now choose ϵ so that $0 < \epsilon \|U\| < 1$ and

(6) $$\lambda < Jf(x)(1 - \epsilon\|U\|)^m < Jf(x)(1 + \epsilon\|U\|)^m < \mu$$

and then select such a positive number δ that

(7) $$2\delta(1 + \epsilon\|U\|) < r\|U\|$$

and

(8) $$|g_s(z) - g_s(x) - L(z - x)| < \epsilon|z - x| \quad \text{whenever} \quad 0 < |z - x| < \delta.$$

Next the classical theorem on continuously differentiable functions with nonvanishing Jacobian yields in view of (4) a number t such that $0 < t < \delta$ and

$$p^*(S) \text{ is an open subset of } E_m,$$

where

$$S = E_m \underset{z}{E}\,[\,|z - x| < t\,] \subset Q_s.$$

Now a typical set $\alpha \in F(x)$ is constructed as follows:

Choose a number ρ for which $0 < \rho \| U \| < t$ and

(9) $$w \in p^*(S) \quad \text{whenever} \quad |w - p(x)| < (1 - \epsilon \| U \|)\rho.$$

Corresponding to this number ρ let

$$\alpha = E_m \underset{z}{E}\,[\,|T(z - x)| \leqq \rho\,],$$
$$\bar{\alpha} = E_n \underset{y}{E}\,[\,|y - g_s(x)| < (1 + \epsilon \| U \|)\rho\,].$$

Clearly $x \in \alpha$, α is closed, $\bar{\alpha}$ is an open sphere of E_n, and $\operatorname{diam} \bar{\alpha} = 2\rho(1 + \epsilon \| U \|) < 2\delta \| U \|^{-1}(1 + \epsilon \| U \|) < r$ by virtue of (7). A moment's thought convinces us that

(10) $$|\alpha| \cdot |\det T| = |E_m \underset{z}{E}\,[\,|z| \leqq \rho\,]|,$$

(11) $$\gamma(\bar{\alpha}) = (1 + \epsilon \| U \|)^m |E_m \underset{z}{E}\,[\,|z| \leqq \rho\,]|.$$

Next we check

(12) $$|z - x| = |U[T(z - x)]| \leqq \| U \| \rho < t < \delta \quad \text{for} \quad z \in \alpha;$$

(13) $$\alpha \subset S \subset Q_s \quad \text{and} \quad \operatorname{diam} \alpha < 2 \| U \| \rho.$$

Evidently $g_s(x) \in \bar{\alpha}$ and the relation $x \neq z \in \alpha$ implies via (12), (8), (5), (12) that

$$\begin{aligned} |g_s(z) - g_s(x)| &< |L(z - x)| + \epsilon |z - x| \leqq |T(z - x)| + \epsilon \| U \| \rho \\ &\leqq \rho(1 + \epsilon \| U \|), \end{aligned}$$

hence $g_s(z) \in \bar{\alpha}$. Accordingly

(14) $$g_s^*(\alpha) \subset \bar{\alpha}.$$

Now let

$$C = E_m \underset{w}{E}\,[\,|w - p(x)| < (1 - \epsilon \| U \|)\rho\,].$$

If $w \in C$ we use (9) to pick $z \in S$ with $p(z) = w$. Since $t < \delta$ we may use (8) and (12) to check

$$\begin{aligned} |p(z) - p(x) - T(z - x)| &= |P[g_s(z)] - P[g_s(x)] - P[L(z - x)]| \\ &= |P[g_s(z) - g_s(x) - L(z - x)]| \\ &\leqq |g_s(z) - g_s(x) - L(z - x)| \\ &\leqq \epsilon |z - x| \leqq \epsilon \| U \| \cdot |T(z - x)|, \end{aligned}$$

$$(1-\epsilon\|U\|)\rho \geqq |p(z)-p(x)| \geqq |T(z-x)| - \epsilon\|U\|\cdot|T(z-x)| = (1-\epsilon\|U\|)\cdot|T(z-x)|,$$

hence $\rho \geqq |T(z-x)|$ and $z\in\alpha$, $w\in p^*(\alpha)$. Thus

$$C \subset p^*(\alpha). \tag{15}$$

From Theorem 3.6 of SA I (see Remark 2.3 of this paper) and the relations (15), (10), (4), (6) we infer

$$\begin{aligned}\Phi[g_s^*(\alpha)] &\geqq |P[g_s^*(\alpha)]| = |p^*(\alpha)| \geqq |C| \\ &= (1-\epsilon\|U\|)^m |E_m \underset{z}{E}[|z|\leqq\rho]| = (1-\epsilon\|U\|)^m|\alpha|\cdot|\det T| \\ &= (1-\epsilon\|U\|)^m Jf(x)|\alpha| > \lambda|\alpha|;\end{aligned}$$

$$\lambda|\alpha| < \Phi[g_s^*(\alpha)]. \tag{16}$$

From (11), (10), (4), (6) we similarly obtain

$$\gamma(\bar{\alpha}) < \mu|\alpha|. \tag{17}$$

At last we use (10) and (13) to check that

$$\frac{|\alpha|}{(\operatorname{diam}\alpha)^m} \geqq \frac{|E_m \underset{z}{E}[|z|\leqq\rho]|}{|\det T|\cdot(2\|U\|\rho)^m} = \frac{|E_m \underset{z}{E}[|z|\leqq 1]|}{|\det T|\cdot(2\|U\|)^m}$$

and observe that the right-hand member of the last equation is a positive number independent of ρ and α.

We combine the last remark with the relations (13), (16), (14), (17) to complete the proof of Part 1.

Part 2. If $X\in K$ and $X\subset B_s$, then

$$\lambda|X| \leqq V_K(f, X, \Phi) \quad\text{and}\quad \Phi[f^*(X)] \leqq \mu|X|.$$

Proof. Let $r>0$.

Choose an open set S such that $X\subset S$ and $|S-X|\leqq r$.

For $x\in X$ define $F(x)$ as in Part 1 and let

$$G(x) = F(x)\,\underset{\alpha}{E}\,[\alpha\subset S],$$

$$U = \sum_{x\in X} G(x).$$

The existence of a countable disjointed family $H\subset U$ for which $|X-\sigma(H)| = 0$ may be deduced from the classical Vitali covering theorem[8] or from the

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[8] S. Saks, *Theory of the integral*, Warsaw, 1937, p. 109.

fact that G is a diametrically regular blanket[9]. Now use Part 1, Lemma 4.1 and (2) to obtain

$$\begin{aligned}\gamma_r[f^*(X)] &= \gamma_r[g_s^*(X)] \leqq \gamma_r\{g_s^*[\sigma(H)]\} + \gamma_r\{g_s^*[X - \sigma(H)]\}\\ &\leqq \sum_{\alpha\in H} \gamma(\bar{\alpha}) + \Phi\{g_s^*[X - \sigma(H)]\}\\ &\leqq \sum_{\alpha\in H} \mu|\alpha| + (10M)^m|X - \sigma(H)|\\ &\leqq \mu|S| \leqq \mu(|X| + r);\end{aligned}$$

$$\begin{aligned}\lambda|X| &\leqq \lambda\sum_{\alpha\in H}|\alpha| \leqq \sum_{\alpha\in H}\Phi[g_s^*(\alpha)] \leqq \sum_{\alpha\in H}\Phi[g_s^*(\alpha X)] + \Phi[g_s^*(\alpha - X)]\\ &\leqq \sum_{\alpha\in H}\Phi[f^*(\alpha X)] + (10M)^m|\alpha - X|\\ &\leqq V_K(f, X, \Phi) + (10M)^m|S - X| \leqq V_K(f, X, \Phi) + (10M)^m r.\end{aligned}$$

Thus

$$\lambda|X| \leqq V_K(f, X, \Phi) + (10M)^m r \quad\text{and}\quad \gamma_r[f^*(X)] \leqq \mu(|X| + r)$$

for every $r>0$. Let $r\to 0$.

Part 3. $\lambda|A| \leqq \int N(f, A, y)d\Phi y \leqq \mu|A|$.

Proof. We use (1), Lemma 4.1 and Theorem 4.1 of SA I to check:

$$0 = |A - B| = \Phi[f^*(A - B)],$$

$$\sum_{s=1}^{\infty}|B_s| = |B| = |A|,$$

$$\sum_{s=1}^{\infty} V_K(f, B_s, \Phi) = \sum_{s=1}^{\infty}\int N(f, B_s, y)d\Phi y = \int\sum_{s=1}^{\infty} N(f, B_s, y)d\Phi y$$

$$= \int N(f, B, y)d\Phi y = \int N(f, A, y)d\Phi y.$$

We shall accordingly complete the proof by showing that

$$\lambda|B_s| \leqq V_K(f, B_s, \Phi) \leqq \mu|B_s| \quad\text{for}\quad s = 1, 2, 3, \cdots. \tag{18}$$

Fixing a positive integer s, we see that the first inequality in (18) follows immediately from Part 2. If, on the other hand, $G\subset K$ and G is a partition of B_s, then

$$\sum_{S\in G}\Phi[f^*(S)] \leqq \mu\sum_{S\in G}|S| = \mu|B_s|$$

by virtue of Part 2. Considering the arbitrary nature of G, we are convinced of the second inequality in (18).

4.4 Lemma. *If f is a function on E_m to E_n; M is a number such that*

$$|f(z) - f(x)| \leqq M|z - x| \quad \textit{for}\quad z\in E_m,\ x\in E_m;$$

(9) Loc. cit. (3), Definition 6.5 and Theorem 6.12.

A is a Lebesgue measurable subset of E_m; f is differentiable at each point of A and $Jf(x)=0$ for $x\in A$; then $\Phi[f^*(A)]=0$.

Proof. We assume, without loss of generality, that $|A|<\infty$ and divide the remainder of the proof into two parts.

Part 1. If $x\in A$ and $r>0$, then there is a closed sphere $\alpha\subset E_m$ and an open connected set $\bar{\alpha}\subset E_n$ such that $x\in\alpha$, $\operatorname{diam}\alpha<r$, $\operatorname{diam}\bar{\alpha}<r$, $f^*(\alpha)\subset\bar{\alpha}$, $\gamma(\bar{\alpha})\leqq r|\alpha|$.

Proof. Let L be the differential of f at x and let h be the function such that

$$h(z) = f(x) + L(z - x).$$

Choose ϵ so that

$$0 < 2\epsilon\{\|L\| + \epsilon\}^{m-1}| E_{m-1} \underset{z}{E} [|z| \leqq 1]| \leqq r| E_m \underset{w}{E} [|w| \leqq 1]|,$$

select ρ so that $0<2\rho\{\|L\|+\epsilon+1\}<r$ and $|f(z)-h(z)|<\epsilon|z-x|$ whenever $0<|z-x|\leqq\rho$.

Denoting by $d(y, Y)$ the distance of the point y from the set Y, we define

$$\alpha = E_m \underset{z}{E} [|z - x| \leqq \rho], \qquad \bar{\alpha} = E_n \underset{y}{E} [d\{y, h^*(\alpha)\} < \epsilon\rho].$$

Of all the properties required for α and $\bar{\alpha}$, only the last is not quite obviously satisfied. In order to prove it, let P be any projection function on E_n to E_m. Denote

$$Q = E_m \underset{w}{E} [w_m = 0].$$

Since $\Delta(L)=Jf(x)=0$, the points $L^1, L^2, \cdots, L^m$ are linearly dependent; hence the same is true of the points $P(L^1), \cdots, P(L^m)$. We may therefore select a distance preserving transformation T on E_m to E_m such that

$$[T:P:h]^*(\alpha) \subset Q.$$

Let

$$B = Q \underset{w}{E} [|w - [T:P:h](x)| \leqq \|L\|\rho],$$
$$C = E_m \underset{w}{E} [|w - [T:P:h](x)| < \{\|L\| + \epsilon\}\rho;\ |w_m| < \epsilon\rho]$$

and successively check the relations

$$h^*(\alpha) \subset E_n \underset{y}{E} [|y - h(x)| \leqq \|L\|\rho], \qquad [T:P:h]^*(\alpha) \subset B,$$
$$[T:P]^*(\bar{\alpha}) \subset E_m \underset{w}{E} [d\{w, [T:P:h]^*(\alpha)\} < \epsilon\rho] \subset E_m \underset{w}{E} [d(w, B) < \epsilon\rho] \subset C.$$
$$\begin{aligned} |P^*(\bar{\alpha})| &= |[T:P]^*(\bar{\alpha})| \leqq |C| \\ &= 2\epsilon\rho\{\|L\| + \epsilon\}^{m-1}\rho^{m-1}\cdot |E_{m-1} \underset{z}{E} [|z| \leqq 1]| \\ &= 2\epsilon\{\|L\| + \epsilon\}^{m-1} \frac{|E_{m-1} \underset{z}{E} [|z| \leqq 1]|}{|E_m \underset{w}{E} [|w| \leqq 1]|} |\alpha| \leqq r|\alpha|. \end{aligned}$$

Thus $|P^*(\bar{\alpha})| \leqq r|\alpha|$ and from the arbitrary nature of P we infer $\gamma(\bar{\alpha}) \leqq r|\alpha|$.

Part 2. $\Phi[f^*(A)]=0$.

Proof. Let $r>0$.

Choose an open set S such that $A\subset S\subset E_m$, $|S|<|A|+r$. Let F be the family of all closed spheres $\alpha\subset S$ which satisfy all the conditions of Part 1. Clearly F covers A in the sense of Vitali and there is a countable disjointed subfamily G of F such that $|A-\sigma(G)|=0$. Use Part 1 and 4.1 to infer

$$\begin{aligned}\gamma_r[f^*(A)] &\leqq \gamma_r\{f^*[\sigma(G)]\} + \gamma_r\{f^*[A-\sigma(G)]\}\\ &\leqq \sum_{\alpha\in G}\gamma(\bar{\alpha}) + \Phi\{f^*[A-\sigma(G)]\} \leqq r\sum_{\alpha\in G}|\alpha| + (5M)^m|A-\sigma(G)|\\ &\leqq r|S| < r\{|A|+r\};\end{aligned}$$

$$\gamma_r[f^*(A)] < r\{|A|+r\}.$$

Let $r\to 0$.

4.5 THEOREM. *If f is a function on E_m to E_n; M is a number such that*

$$|f(z)-f(x)| \leqq M|z-x| \quad \textit{for} \quad z\in E_m,\ x\in E_m;$$

T is a Lebesgue measurable subset of E_m; then

$$\int N(f,T,y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Let A be the set of those points of T at which f is not differentiable. From Rademacher's theorem[10] we know that $|A|=0$. Hence $\Phi[f^*(A)]=0$ by 4.1. Denote

$$B=(T-A)\mathop{E}_{x}[Jf(x)=0],\qquad C=(T-A)\mathop{E}_{x}[Jf(x)>0],$$

and observe that the theorem follows immediately from the following

Statement. If $t>1$, then

$$t^{-1}\int_T Jf(x)dx \leqq \int N(f,T,y)d\Phi y \leqq t\int_T Jf(x)dx.$$

Proof. Let α be the set of all integers. For $j\in\alpha$ let

$$C_j = C\mathop{E}_{x}[t^j < Jf(x) \leqq t^{j+1}],$$

and infer from 4.3, taking $\lambda=t^j$ and letting $\mu\to t^{j+1}+$, that

$$t^j|C_j| \leqq \int N(f,C_j,y)d\Phi y \leqq t^{j+1}|C_j|$$

(10) H. Rademacher, *Über partielle und totale Differenzierbarkeit*, I, Math. Ann. vol. 79 (1919).

because C_j is a Borel set. Note that

$$T = A + B + \sum_{j\in\alpha} C_j,$$

and accordingly

$$t^{-1}\int_T Jf(x)dx = t^{-1}\int_A Jf(x)dx + t^{-1}\int_B Jf(x)dx + \sum_{j\in\alpha} t^{-1}\int_{C_j} Jf(x)dx$$

$$\leqq \sum_{j\in\alpha} t^j |C_j| \leqq \sum_{j\in\alpha}\int N(f, C_j, y)d\Phi y = \int \sum_{j\in\alpha} N(f, C_j, y)d\Phi y$$

$$= \int N(f, C, y)d\Phi y = \sum_{j\in\alpha}\int N(f, C_j, y)d\Phi y \leqq \sum_{j\in\alpha} t^{j+1}|C_j|$$

$$\leqq \sum_{j\in\alpha} t\int_{C_j} Jf(x)dx \leqq t\int_T Jf(x)dx;$$

$$t^{-1}\int_T Jf(x)dx \leqq \int N(f, C, y)d\Phi y \leqq t\int_T Jf(x)dx.$$

But $\Phi[f^*(A+B)] \leqq \Phi[f^*(A)] + \Phi[f^*(B)] = 0$ in view of our remark at the beginning of the proof and of Lemma 4.4. Thus

$$N(f, C, y) = N(f, T, y)$$

for Φ almost all y in E_n and we complete the proof by substitution in the last integral inequality.

5. **Approximately differentiable surfaces.**

5.1 THEOREM. *If f is a Lebesgue measurable function on E_m to E_n, T is a Lebesgue measurable subset of E_m and if there are sets $T_1, T_2, T_3, \cdots$ and numbers $M_1, M_2, M_3, \cdots$ such that*

$$T = \sum_{j=1}^{\infty} T_j$$

and

$$|f(z) - f(x)| \leqq M_j|z - x| \quad \textit{whenever} \quad z \in T_j,\ x \in T_j,$$

then

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Let C be the set of points of approximate continuity of f. For each positive integer j let D_j be the set of points of density of T_j and define

$$S_j = TCD_j, \qquad A_j = S_j - \sum_{i=1}^{j-1} S_i.$$

We note that S_j is a Lebesgue measurable set with

$$|T_j - S_j| = |T_j - TCD_j| = |T_j - CD_j|$$
$$\leqq |T_j - C| + |T_j - D_j| = 0,$$
$$(1) \qquad |T_j - S_j| = 0 \quad \text{for} \quad j = 1, 2, 3, \cdots.$$

The remainder of the proof is divided into four parts.

Part 1. If $x \in S_j$, $z \in S_j$, then $|f(z)-f(x)| \leqq M_j|z-x|$.

Proof. Select a measurable set A with density 1 at x such that f is continuous relative to A at x. Since A is measurable it is easily seen(11) that x is a point of density of the set AT_j, even though T_j may be nonmeasurable. Hence there are points $u^1, u^2, u^3, \cdots$ such that

$$\lim_{k\to\infty} u^k = x, \qquad u^k \in AT_j \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

It follows that

$$\lim_{k\to\infty} f(u^k) = f(x), \quad u^k \in T_j \qquad \text{for} \quad k = 1, 2, 3, \cdots.$$

Applying the same reasoning to z we find points $v^1, v^2, v^3, \cdots$ for which

$$\lim_{k\to\infty} v^k = z, \qquad \lim_{k\to\infty} f(v^k) = f(z), \qquad v^k \in T_j \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Accordingly

$$|f(z) - f(x)| = |\lim_{k\to\infty} f(v^k) - \lim_{k\to\infty} f(u^k)| = \lim_{k\to\infty} |f(v^k) - f(u^k)|$$
$$\leqq M_j \limsup_{k\to\infty} |v^k - u^k| = M_j|z - x|.$$

Part 2. If j is a positive integer, then

$$\int N(f, A_j, y)d\Phi y = \int_{A_j} Jf(x)dx.$$

Proof. In view of Part 1 there exists(12) a function g on E_m to E_n such that g satisfies a Lipschitz condition on E_m and

$$(2) \qquad g(x) = f(x) \quad \text{for} \quad x \in S_j.$$

It follows from Theorem 4.5 that

$$(3) \qquad \int N(g, A_j, y)d\Phi y = \int_{A_j} Jg(x)dx.$$

(11) The relation $|XA| + |XT_j| = |X(A+T_j)| + |XAT_j|$ holds for every $X \subset E_m$. See C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1927, p. 252.

(12) H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. vol. 36 (1934). Apply the formula given in the last footnote on page 63 to each of the n coordinate functions.

But $A_j \subset S_j$; hence (2) implies

$$N(g, A_j, y) = N(f, A_j, y) \quad \text{for} \quad y \in E_n. \tag{4}$$

Moreover at each point of density of S_j at which g is differentiable, the differential of g is the approximate differential of f. Accordingly

$$Jg(x) = Jf(x) \text{ for almost all } x \text{ in } S_j. \tag{5}$$

Substitute (4) and (5) in (3) to establish Part 2.

Part 3. If j is a positive integer, then $\Phi[f^*(T_j - S_j)] = 0$.

Proof. Remember that f satisfies a Lipschitz condition on T_j and apply (1) and 4.1.

Part 4. $\int N(f, T, y)d\Phi y = \int_T Jf(x)dx$.

Proof. Let $S = \sum_{j=1}^{\infty} S_j$; note that $A_1, A_2, A_3, \cdots$ are disjoint and

$$S = \sum_{j=1}^{\infty} A_j, \qquad (T - S) \subset \sum_{j=1}^{\infty} (T_j - S_j). \tag{6}$$

Hence (1) and Part 3 imply

$$|T - S| = \Phi[f^*(T - S)] = 0, \tag{7}$$

$$N(f, T, y) = N(f, S, y) \quad \text{for } \Phi \text{ almost all } y \text{ in } E_n. \tag{8}$$

We now use (7), Part 2, (6), (8) to conclude

$$\begin{aligned} \int_T Jf(x)dx &= \int_S Jf(x)dx = \sum_{j=1}^{\infty} \int_{A_j} Jf(x)dx \\ &= \sum_{j=1}^{\infty} \int N(f, A_j, y)d\Phi y = \int \sum_{j=1}^{\infty} N(f, A_j, y)d\Phi y \\ &= \int N(f, S, y)d\Phi y = \int N(f, T, y)d\Phi y. \end{aligned}$$

5.2 THEOREM. *If f is a Lebesgue measurable function on E_m to E_n, T is a Lebesgue measurable subset of E_m and*

$$\limsup_{z \to x} \text{ap} \frac{|f(z) - f(x)|}{|z - x|} < \infty \quad \textit{for every} \quad x \in T,$$

then

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Let

$$K_0 = E_m \underset{w}{E} [|w| \leq 1],$$

$$K_1 = E_m \underset{w}{E} [|w - (1, 0, \cdots, 0)| \leq 1]$$

and denote

$$\alpha = \frac{|K_0 K_1|}{|K_0| + |K_1|}.$$

Since $K_0 K_1$ has interior points, we know that $\alpha > 0$.

For each positive integer j, we let T_j be the set of all points x for which $x \in T$, $|f(x)| \leqq j$, and $0 < r \leqq j^{-1}$ implies

$$\left| \underset{w}{E} [\,|w - x| \leqq r,\ |f(w) - f(x)| \leqq j|w - x|\,] \right| > (1 - \alpha)\left| \underset{w}{E} [\,|w - x| \leqq r] \right|.$$

Evidently

$$T = \sum_{j=1}^{\infty} T_j$$

and use of 5.1 will complete the proof as soon as we verify the following

Statement. If $x \in T_j$, $z \in T_j$, then

$$|f(z) - f(x)| \leqq 2j^2 |z - x|.$$

Proof. If $|z - x| > j^{-1}$, we have $|f(z) - f(x)| \leqq 2j < 2j^2|z - x|$. We henceforth assume $|z - x| = r \leqq j^{-1}$ and define

$$C_0 = \underset{w}{E} [\,|w - x| \leqq r], \qquad C_1 = \underset{w}{E} [\,|w - z| \leqq r];$$

note that

$$\alpha = \frac{|C_0 C_1|}{|C_0| + |C_1|}$$

and let

$$B_0 = C_0 \underset{w}{E} [\,|f(w) - f(x)| \leqq j|w - x|\,],$$
$$B_1 = C_1 \underset{w}{E} [\,|f(w) - f(z)| \leqq j|w - z|\,].$$

Since f is a measurable function we know that B_0 and B_1 are measurable sets and

$$\begin{aligned} |B_0 B_1| &= |B_0| + |B_1| - |B_0 + B_1| \\ &> (1 - \alpha)|C_0| + (1 - \alpha)|C_1| - |C_0 + C_1| \\ &= (1 - \alpha)\{|C_0| + |C_1|\} - \{|C_0| + |C_1| - |C_0 C_1|\} \\ &= |C_0 C_1| - \alpha\{|C_0| + |C_1|\} = 0. \end{aligned}$$

Thus $|B_0 B_1| > 0$ and we can pick a point $w \in B_0 B_1$. Then

$$\begin{aligned} |f(z) - f(x)| &\leqq |f(z) - f(w)| + |f(w) - f(x)| \leqq j|z - w| + j|z - x| \\ &= 2jr = 2j|z - x| \leqq 2j^2|z - x|. \end{aligned}$$

5.3 THEOREM. *If f is a continuous function on E_m to E_n, T is an analytic* *subset of* E_m *and the approximate partial derivatives of* f *exist (finite) at almost all points of* T, *then*

$$\int_T Jf(x)dx \leqq \int N(f, T, y)d\Phi y.$$

Proof. Let S be the subset of T on which f is approximately differentiable. From Stephanoff's theorem([13]) we see that $|T-S|=0$ and combine this with 5.2 to infer

$$\int_T Jf(x)dx = \int_S Jf(x)dx = \int N(f, S, y)d\Phi y.$$

But $N(f, S, y) \leqq N(f, T, y)$ for $y \in E_n$ and completion of the proof depends merely on showing that the last expression is Φ measurable. For this purpose we use Theorem 4.1 of SA I, taking $F=$ the family of all analytic subsets of T.

5.4 *Remark. Suppose* $\int_T Jf(x)dx < \infty$ *in addition to the hypotheses of* 5.3. *Then equality holds in the conclusion of that theorem if and only if*

$$\Phi[f^*(X)] = 0 \quad \textit{whenever} \quad X \subset T, \ |X| = 0.$$

6. Two-dimensional surfaces in n-dimensional space.

6.1 THEOREM. *If* f *is a Lebesgue measurable function on* E_2 *to* E_n $(2 \leqq n)$; T *is a Lebesgue measurable subset of* E_2; *corresponding to each* $x \in T$ *there are three distinct points* $\bar{x}^1, \bar{x}^2, \bar{x}^3$ *of* E_2 *such that* $|\bar{x}^j| = 1$ *and*

$$\limsup_{t \to 0+} \left| \frac{f(x+t\bar{x}^j) - f(x)}{t} \right| < \infty \textit{ for } j = 1, 2, 3;$$

then

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Let $C = E_2 E_z[|z| = 1]$.

Let W be the set of all matrices whose columns are three distinct points of C. For each $x \in T$, the points $\bar{x}^1, \bar{x}^2, \bar{x}^3$ are evidently the columns of the matrix $\bar{x} \in W$. Each matrix $w \in W$ determines a subdivision of C into three arcs as follows: $C_1(w)$ leads from w^2 to w^3; $C_2(w)$ from w^3 to w^1; $C_3(w)$ from w^1 to w^2. Further we denote by $\delta(w)$ the smallest of the distances from w^j to $C_j(w)$, where $j = 1, 2, 3$, and observe that $\delta(w) > 0$ for $w \in W$.

Now let D be a countable subset of C which is dense in C and define

$$Z = W \underset{w}{E} [w^j \in D \text{ for } j = 1, 2, 3].$$

We write $a \cdot b = a_1 b_1 + a_2 b_2$ for $a \in E_2$, $b \in E_2$ and note that $|\bar{x}^i \cdot \bar{x}^j| < 1$ whenever $x \in T$; $i = 1, 2, 3$; $j = 1, 2, 3$; $i \neq j$.

([13]) Loc. cit. ([7]), p. 300. The extension to m dimensions offers no difficulty.

For each positive integer k and each matrix $w \in Z$, we let T_k^w be the set of all points x for which:

$$x \in T; \qquad |f(x)| \leqq k; \qquad \delta(\bar{x}) > 4k^{-1}.$$

$$|f(x + t\bar{x}^j) - f(x)| \leqq kt \quad \text{whenever} \quad 0 \leqq t \leqq 2k^{-1}, j = 1, 2, 3.$$

$$1 - x^i \cdot x^j > 3k^{-1} \quad \text{whenever} \quad i = 1, 2, 3; j = 1, 2, 3; i \neq j.$$

$$|\bar{x}^j - w^j| \leqq k^{-1} \quad \text{for} \quad j = 1, 2, 3.$$

Since Z is countable and

$$T = \sum_{k=1}^{\infty} \sum_{w \in Z} T_k^w$$

our theorem is a consequence of 5.1 and the following

Statement. If $x \in T_k^w$ and $z \in T_k^w$, then

$$|f(z) - f(x)| \leqq 2k^3 |z - x|.$$

Proof. Let $r = k^{-1}$. In case $|z - x| \geqq r^2$, we have

$$|f(z) - f(x)| \leqq 2k = 2k^3 r^2 \leqq 2k^3 |z - x|$$

and we henceforth assume

(1) $$|z - x| < r^2.$$

We define

$$K = \underset{u}{E}\,[|u - x| = r],$$

$$S_j = \underset{u}{E}\left[|u - x| \leqq r, \frac{u - x}{|u - x|} \in C_j(\bar{x})\right], \qquad K_j = KS_j \quad \text{for} \quad j = 1, 2, 3;$$

and let L_j be the line segment joining x and $x + r\bar{x}^j$. Since (1) implies $z \in \sum_{j=1}^{3} S_j$, we may, without loss of generality, assume $z \in S_3$.

The half ray issuing from z with direction $\bar{z}^3$ intersects the boundary of S_3 in a point u; that is, a number $\lambda \geqq 0$ can be found such that

(2) $$z + \lambda\bar{z}^3 = u \in (L_1 + L_2 + K_3).$$

We have $\lambda = |\lambda\bar{z}^3| = |u - z| \leqq 2r$ since z and u are inside K; hence

(3) $$0 \leqq \lambda \leqq 2r.$$

We shall prove next (by contradiction) that

(4) $$u \in L_1 + L_2.$$

By virtue of (2), the denial of (4) implies $u \in K_3$. From the definition of K_3 we infer

$$|u - x| = r, \qquad (u - x)/r \in C_3(x),$$

and use the definitions of $\delta(\bar{x})$ and T_k^w as well as the relations (2), (1) to obtain

$$\begin{aligned}\delta(\bar{x}) &\leqq |(u - x)/r - \bar{x}^3| = k|z + \lambda\bar{z}^3 - x - r\bar{x}^3| \\ &\leqq k\{|z - x| + |\lambda\bar{z}^3 - r\bar{z}^3| + |r\bar{z}^3 - rw^3| + |rw^3 - r\bar{x}^3|\} \\ &< k\{r^2 + |\lambda - r| + r^2 + r^2\} = 3r + k|\lambda - r| \\ &= 3r + k||u - z| - |u - x|| \leqq 3r + k|x - z| < 3r + r < \delta(\bar{x});\end{aligned}$$

hence $\delta(\bar{x}) < \delta(\bar{x})$, which is false. Thus we have verified (4) and we may, without loss of generality, assume

$$u \in L_1.$$

We infer the existence of a number μ such that

$$0 \leqq \mu \leqq r, \qquad x + \mu\bar{x}^1 = u. \tag{5}$$

Use (2), (5) and the definition of T_k^w to compute

$$z - x = \mu\bar{x}^1 - \lambda\bar{z}^3, \qquad (z - x)\cdot(\bar{x}^1 - \bar{z}^3) = (\mu + \lambda)(1 - \bar{z}^3\cdot\bar{x}^1),$$

$$1 - \bar{z}^3\cdot\bar{x}^1 = 1 - \bar{x}^3\cdot\bar{x}^1 + (\bar{x}^3 - \bar{z}^3)\cdot\bar{x}^1 \geqq 3r - |\bar{x}^3 - \bar{z}^3| \geqq r,$$

$$\lambda + \mu \leqq \frac{|z - x|\cdot|\bar{x}^1 - \bar{z}^3|}{1 - \bar{z}^3\cdot\bar{x}^1} \leqq (2/r)|z - x|;\ \mu + \lambda \leqq 2k|z - x|.$$

The last relation combines with (2), (5), (3) and the definition of T_k^w to give us the result:

$$\begin{aligned}|f(z) - f(x)| &\leqq |f(z) - f(z + \lambda\bar{z}^3)| + |f(x + \mu\bar{x}^1) - f(x)| \\ &\leqq k\lambda + k\mu = k(\lambda + \mu) \leqq 2k^2|z - x| \leqq 2k^3|z - x|.\end{aligned}$$

6.2 *Remark.* It is easy to show by an example that the three directions associated with each point in 6.1 cannot be replaced by two directions.

6.3 THEOREM. *If the Lebesgue measurable function f on E_2 to E_n $(2 \leqq n)$ has finite partial derivatives at each point of the Lebesgue measurable set T, then*

$$\int N(f, T, y)d\Phi y = \int_T Jf(x)dx.$$

Proof. Take $\bar{x}^1 = (1, 0)$; $\bar{x}^2 = (0, 1)$; $\bar{x}^3 = (-1, 0)$ for $x \in T$ and apply 6.1.

7. Functions on the line to the line.

7.1 *Definition.* If f is a function on E_1 to E_1, then D^+f is the function such that

$$D^+f(x) = \limsup_{z \to x+} \frac{f(z) - f(x)}{z - x}.$$

7.2 THEOREM. *If f is a Lebesgue measurable function on E_1 to E_1, T is a Lebesgue measurable set of numbers and*

$$|D^+f(x)| < \infty \quad \text{for} \quad x \in T,$$

then

$$\int N(f, T, y)dy = \int_T |D^+f(x)| dx.$$

Proof. From Theorem 9.9 of Saks[14], chap. 9, we infer the existence of a set $S \subset T$ such that

$$|T - S| = 0, \qquad Jf(x) = |D^+f(x)| \quad \text{for} \quad x \in S.$$

Hence Theorem 5.2 of this paper yields

$$\int N(f, S, y)dy = \int_S Jf(x)dx = \int_S |D^+ f(x)| dx = \int_T |D^+ f(x)| dx.$$

But Theorem 4.6 of Saks[14], chap. 9, tells us

$$|f^*(T - S)| = 0$$

and we complete the proof by observing that the last relation implies $N(f, S, y) = N(f, T, y)$ for almost all numbers y.

([14]) Loc. cit. ([7]).

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF.